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ON THE CONVEXITY OF THE WEAKLY COMPACT CHEBYSHEV SETS IN BANACH SPACES

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ABSTRACT

A sufficient condition for a Banach space X is given so that every weakly compact Chebyshev subset of X is convex. For this purpose a class broader than that of smooth Banach spaces is defined. In this way a former result of A. Brøndsted and A. L. Brown is partially extended in every finite dimensional normed linear space and a known result in Hilbert spaces is proved in a different way.

1. Introduction

A subset M of a normed linear space X is called a **Chebyshev set** if to each point x of X there exists a unique point of M that is nearest to x. It is well known that every closed convex subset of a Hilbert space is a Chebyshev set. A Banach space X is said to have the **Chebyshev property** if every Chebyshev subset of X is convex. It is known that every n-dimensional Euclidean space (L. N. H. Bunt [6], T. S. Motzkin [14], B. Jessen [11]) or, more generally, a smooth finite dimensional normed linear space (H. Busemann [7]) has the Chebyshev property. Also, there are examples of non-smooth spaces which have the Chebyshev property. In particular A. Brøndsted ([3], [4]) constructed non-smooth n-dimensional normed linear spaces with the Chebyshev property for every $n \geq 3$. He also proved that if $n \leq 3$, an n-dimensional normed linear space has the Chebyshev

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property if and only if every exposed point of its unit ball is also a smooth point. It turns out that if $n = 2$ the latter property is equivalent to the smoothness of the unit ball. The spaces that A. Brøndsted constructed have the above property. He also generalized such constructions for non-smooth reflexive Banach spaces. Later, A. L. Brown extended Brøndsted's characterization of spaces with the Chebyshev property for every normed linear space X with $\dim X = 4$. It is unknown whether the same characterization holds for higher dimensions.

In the infinite dimensional case N. V. Efimov and S. B. Stechkin ([9]) proved that every weakly closed Chebyshev subset of a smooth and uniformly rotund Banach space is convex (see also [17]) and L. P. Vlasov ([16] or [17]) proved that every boundedly compact Chebyshev subset of a smooth Banach space is convex. Similar conclusions were also obtained by V. Klee ([13]).

In the present paper we are concerned mainly with the convexity of the weakly compact Chebyshev sets. Our results partially extend those of A. Brøndsted and A. L. Brown in every finite dimensional linear space (see Corollary 1.2) and also give a unified approach to all the above mentioned results in the case of weakly compact Chebyshev sets. We begin with the following definition:

Definition 1.1: Let X be a Banach space with unit ball B_X and $SM(B_X)$ the set of all functionals of X^* which attain their norm at a smooth point of B_X. Then X will be called **almost smooth** if $SM(B_X)$ is dense in X^*.

We note that by the Bishop–Phelps Theorem (cf. [2]), every smooth Banach space is almost smooth. Also, as is known, for every finite dimensional normed space X, the set of functionals of X^* which expose points of B_X is dense in X^* (see Theorem 2.2.9 of [15]). Therefore, if every exposed point of B_X is also a smooth point then X is almost smooth. Moreover, if X is a reflexive space or, more generally, a Banach space with the RNP and every strongly exposed point of the unit ball is a smooth point, then by the Phelps–Bourgain Theorem (cf. [2]) X is almost smooth. Hence A. Brøndsted's constructions are finite and infinite dimensional examples of non-smooth but almost smooth Banach spaces.

The following is the main result of the present paper:

THEOREM 1.1: *If X is an almost smooth Banach space then every weakly compact Chebyshev subset of X is convex.*

As an immediate consequence of Theorem 1.1 we get the following result which is directly related to that of A. Brøndsted and A. L. Brown.

COROLLARY 1.2: *If X is a finite dimensional normed linear space such that every exposed point of B_X is a smooth point then every bounded Chebyshev subset of*

X is convex.

The proof of Theorem 1.1 is based on the concept of the n-extreme points and the technique introduced in [12]. We are unable, in general, to extend these results in the case of unbounded Chebyshev sets. However, in the special case of a Hilbert space we prove the following known result as a consequence of Theorem 1.1 and the fact that a weakly closed set in a Hilbert space is a Chebyshev set if and only if its intersections with the balls centered at some point of it are Chebyshev sets (see Lemma 2.6).

THEOREM 1.3: *If H is a Hilbert space then every weakly closed Chebyshev set of H is convex.*

Of course Theorem 1.3 is a Corollary of the relative work of N. V. Efimov and S. B. Stechkin, V. Klee and E. Asplund ([1]). It remains an open problem if a Hilbert space has the Chebyshev property.

Finally, in the last part of the paper we prove some similar results for weakly* compact Chebyshev subsets of a dual almost smooth space and for bounded Chebyshev sets with the RNP.

2. Proofs of Theorems 1.1 and 1.3

We begin with the definition of the notion of n-extreme points. Let X be a Banach space, $L \subset X$ be a closed convex set and suppose that $x \in L$. Then x is called an n-**extreme** point of L ($n \in \mathbb{N}$) if x is in the relative interior of no $(n+1)$-dimensional convex subset of L (see also [15]). The set of all n-extreme points of L is denoted by $Ex_n(L)$. We also define $E(L) = \bigcup_{n=0}^{\infty} Ex_n(L)$. Evidently, if $n = 0$ we get the usual notion of an extreme point of L. Also, if $m < n$ then $Ex_m(L) \subset Ex_n(L)$. Let us define by $\mathcal{F}_x$ the set of closed faces of L that contain x. Since $L \in \mathcal{F}_x$, $\mathcal{F}_x \neq \emptyset$. We set $A_x = \bigcap_{F \in \mathcal{F}_x} F$. Evidently A_x is the smallest closed face of L that contains x. It is easy to see that $x \in Ex_n(L)$ iff $\dim A_x \leq n$ and $x \in \operatorname{relint} A_x$.

The following Lemma has been proved in [12] but we repeat the proof here for the sake of completeness.

LEMMA 2.1: *Let L be a weakly compact convex subset of a Banach space X. Then $L = \overline{E(L)}^{w}$.*

Proof: If L is finite dimensional, then evidently $L = E(L) = Ex_m(L)$ where $m = \dim L$. If L is infinite dimensional let V be a weakly open basic neighborhood of a point $x_0 \in L$. Since $E(L) = \bigcup_{n=0}^{\infty} Ex_n(L)$ it suffices to prove that there exists an $m \in \mathbb{N}$ which depends on V such that $V \cap Ex_m(L) \neq \emptyset$. Suppose that V is determined by the functionals $f_1, \dots, f_k \in X^*$. We set $F = x_0 + \bigcap_{i=1}^k \operatorname{Ker} f_i$. Then F is an m-codimensional affine subspace of X for some $m \leq k$, $x_0 \in F$ and $F \subset V$. Evidently, $F \cap L$ is a non-empty compact convex subset of X and so $Ex_0(F \cap L) \neq \emptyset$ by Krein–Milman's Theorem. Let $z \in Ex_0(F \cap L)$. It is enough to prove that $z \in Ex_m(L)$. Suppose that $z \notin Ex_m(L)$. Then there exists an $(m+1)$-dimensional closed ball B_{m+1} with center at z such that $B_{m+1} \subset L$. Since F is a flat of codimension m and $z \in F$ being at the same time the center of an $(m+1)$-dimensional closed ball B_{m+1}, there exists a segment $[x_1, x_2] \subset F \cap B_{m+1} \subset F \cap L$ such that z is the midpoint of $[x_1, x_2]$. Hence $z \notin Ex_0(F \cap L)$, which is a contradiction. ∎

Proof of Theorem 1.1: Let X be an almost smooth Banach space and M a weakly compact Chebyshev subset of X. Let also $L = \overline{\operatorname{conv}}(M)$. Then L is weakly compact by Krein's Theorem (see Theorem 80 in [10]). By Lemma 2.1, $L = \overline{E(L)}^w$, and since $E(L) = \bigcup_{n=0}^{\infty} Ex_n(L)$ and M is weakly compact it is enough to prove that $Ex_n(L) \subset M$ for every $n \in \mathbb{N}$. By Milman's Theorem (see Theorem 74 in [10]) we have that $Ex_0(L) \subset M$.

So let us suppose that for some $n \geq 1$ there exists a $x \in Ex_n(L) \setminus Ex_{n-1}(L)$ such that $x \notin M$ whereas $Ex_{n-1}(L) \subset M$. We can suppose that $x = 0$. So there exists a unique face A_0 of L such that $\dim A_0 = n$ and $0 \in \operatorname{relint}(A_0)$. It is easy to see that the relative boundary ∂A_0 of A_0 is a subset of $Ex_{n-1}(L)$. Therefore $\partial A_0 \subset M$.

Three kinds of projections will be used. Let $F_n = \operatorname{span}[A_0]$. Since F_n is a finite dimensional subspace of X there exists a closed subspace $F_n^\perp$ of X such that $X = F_n^\perp \oplus F_n$ and so we can define the projection $P_{F_n}\colon X \to F_n$. Since F_n is finite dimensional, P_{F_n} is weak-norm continuous. We also define $P_{\partial A_0}\colon F_n \setminus \{0\} \to \partial A_0$ where $P_{\partial A_0}(x) = \{tx : t > 0\} \cap \partial A_0$, for every $x \in F_n \setminus \{0\}$. The mapping $P_{\partial A_0}$ is well defined and continuous. Finally, let us also consider the mapping $P_M\colon X \to M$ where $P_M(x)$ is the unique nearest point of x in M. This mapping is norm-weak continuous. Indeed, let $(x_n)_n$ be a sequence which strongly converges to a $x \in X$. Then for every n,

$$| \, \|x - P_M(x)\| - \|x_n - P_M(x_n)\| \, | = |d(x, M) - d(x_n, M)| \leq \|x - x_n\|$$

and

$$| \, \|x - P_M(x_n)\| - \|x_n - P_M(x_n)\| \, | \leq \|x - x_n\| .$$

Therefore,

$$| \, \|x - P_M(x)\| - \|x - P_M(x_n)\| \, | \leq 2 \|x - x_n\|$$

so $\|x - P_M(x_n)\| \to \|x - P_M(x)\|$.

Since M is weakly compact it is enough to prove that $P_m(x)$ is the unique limit point of the set $\{P_M(x_n) : n \in \mathbb{N}\}$. So let w be a limit point of this set. By the weak lower semicontinuity of the norm $\|x - w\| \leq \liminf \|x - P_M(x_n)\| = \|x - P_M(x)\|$. Since $w \in M$ this can happen only if $w = P_M(x)$.

With respect to M and $F_n^{\perp}$ two cases are distinguished:

CASE 1: $M \cap F_n^{\perp} = \emptyset$. We define the mapping $R\colon A_0 \to \partial A_0$ where $R(x) = P_{\partial A_0}(P_{F_n}(P_M(x)))$, $x \in A_0$. Since $P_M(x) \in M$ and $M \cap F_n^{\perp} = \emptyset$, $P_M(x) \notin F_n^{\perp}$ and so $P_{F_n}(P_M(x)) \in F_n \setminus \{0\}$, for every $x \in A_0$. Therefore R is a well defined mapping. By the continuity properties of the composed functions P_M, P_{F_n} and $P_{\partial A_0}$ we conclude that R is norm-norm continuous. Since $\partial A_0 \subset M$, $R(x) = x$ for every $x \in \partial A_0$ and so R is a retraction of the convex n-dimensional closed set A_0 onto its relative boundary ∂A_0, which is a contradiction by Brouwer's Theorem (cf. [8]).

CASE 2: $M \cap F_n^{\perp} \neq \emptyset$.

LEMMA 2.2: *There exists a $g \in X^*$ and an $a > 0$ such that $M \cap F_n^{\perp} \subset g^{-1}([a, +\infty)) \cap L \cap F_n^{\perp}$.*

Proof: Let us consider the weakly compact convex set $L \cap F_n^{\perp}$. Since $F_n^{\perp}$ is complementary to $F_n = \operatorname{span}[A_0]$ and A_0 is a face that contains 0, it is easy to see that $0 \in Ex_0\left(L \cap F_n^{\perp}\right)$. Since $0 \notin M$ there exists a weakly open nbd V of 0 such that $V \cap M = \emptyset$. By Choquet's Lemma (see Lemma 73 in [10]) there exists $g \in X^*$ and a $a > 0$ such that the slice $S = \left\{x \in L \cap F_n^{\perp} : g(x) < a\right\}$ is contained in $V \cap L \cap F_n^{\perp}$ and so $M \cap F_n^{\perp} \cap S = \emptyset$. Hence $M \cap F_n^{\perp} \subset g^{-1}([a, +\infty)) \cap L \cap F_n^{\perp}$.
∎

We set $A_1 = g^{-1}([a, +\infty)) \cap L \cap F_n^{\perp}$. The set A_1 is a non-empty weakly compact convex subset of X. Clearly $A_0 \cap A_1 = \emptyset$ and by the previous Lemma $M \cap F_n^{\perp} \subset A_1$. We also set $B(x, r) = \{y \in X : \|x - y\| \leq r\}$ and $\mathring{B}(x, r) = \{y \in X : \|x - y\| < r\}$ for $x \in X$ and $r \geq 0$.

LEMMA 2.3: *There exists a $z_0 \in X$ and an $r_o > 0$ such that $A_0 \subset B(z_0, r_0)$ and $A_1 \cap B(z_0, r_0) = \emptyset$.*

Proof: The sets A_0, A_1 are weakly compact disjoint convex subsets of X and so by the Hahn–Banach Theorem there exists an $f \in X^*$ and $c > 0$ such that $\max_{x \in A_0} f(x) < c < \min_{x' \in A_1} f(x')$. Since A_0, A_1 are bounded and $SM(B_X)$ is dense in X^* we can suppose that $f \in SM(B_X)$. We can also suppose that $\|f\| =$

1. So there exists a smooth point $x_0 \in S_X$ such that $f(x_0) = 1$. Let $z_n = -ncx_0$, $r_n = (n+1)c$, $n \in \mathbb{N}$. We observe that the balls $B_n = B(-ncx_0, (n+1)c)$ are supported at the smooth point cx_0 by the hyperplane $H = f^{-1}(c)$. As is well known (see Lemma 3 in [17]) $\bigcup_{n\in\mathbb{N}} \overset{\circ}{B}_n = f^{-1}((-\infty, c))$.

We will prove that there exists a $n_0 \in \mathbb{N}$ such that the open ball $\overset{\circ}{B}_{n_0}$ contains A_0. Indeed, in the opposite case there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of points of A_0 such that for every $n \in \mathbb{N}$, $y_n \notin \overset{\circ}{B}_n$. Since A_0 is compact we can suppose that $(y_n)_{n\in\mathbb{N}}$ converges to a point $y_0 \in A_0$. Evidently $y_0 \notin \overset{\circ}{B}_n$ for every $n \in \mathbb{N}$ and so $f(y_0) \geq c$, which is a contradiction because $f(x) < c$ for every $x \in A_0$. Finally, since $B(z_n, r_n) \subset f^{-1}((-\infty, c])$, $A_1 \cap B(z_n, r_n) = \emptyset$ for every $n \in \mathbb{N}$. We set $z_0 = z_{n_0}$ and $r_0 = r_{n_0}$. ∎

Let us define $D = \{x \in X : x = tz + (1-t)z_0, z \in \partial A_0, t \in [0,1]\}$.

Lemma 2.4: *For every $x \in D$ we have that $P_M(x) \notin F_n^{\perp}$.*

Proof: Let $x \in D$. Then there exists a $z \in \partial A_0$, $t \in [0,1]$ such that $x = tz + (1-t)z_0$. Since $\partial A_0 \subset M$, $P_M(x) \in B(x, ||x-z||) \subset B(z_0, ||z_0 - z||) \subset B(z_0, r_0)$. If $P_M(x) \in F_n^{\perp}$ then $P_M(x) \in F_n^{\perp} \cap M \subset A_1$ as well. But then $B(z_0, r_0) \cap A_1 \neq \emptyset$, a contradiction. ∎

We define the mapping R': $D \to \partial A_0$ where $R'(x) = P_{\partial A_0}(P_{F_n}(P_M(x)))$, $x \in D$. By the previous Lemma, $P_{F_n}(P_M(x)) \in F_n \setminus \{0\}$ and so R' is well defined. As in case 1 we can verify that R' is a norm-norm continuous retraction of D onto ∂A_0. But D is a star convex set with z_0 as a center and so it is contractible at the point z_0. Therefore ∂A_0 should be contractible at the point $R'(z_0)$, which is a contradiction by Brouwer's Theorem. The proof of Theorem 1.1 is complete. ∎

Proof of Theorem 1.3: Let us denote by $\langle \cdot, \cdot \rangle$ the inner product in H.

Lemma 2.5: *Let H be a Hilbert space and $x, y \in H$. Then for every $\lambda \in (0,1)$ we have that $B(x, ||x-y||) \cap B(0, ||y||) \subset B(\lambda x, ||\lambda x - y||) \cap B(0, ||y||)$.*

Proof: It is enough to prove that if $w \in H$ such that $||x-w|| \leq ||x-y||$ and $||w|| \leq ||y||$, then $||\lambda x - w|| \leq ||\lambda x - y||$ or, equivalently,

$$-2\langle x, w\rangle \leq -2\langle x, y\rangle + \frac{1}{\lambda}(\langle y, y\rangle - \langle w, w\rangle) \quad \text{for every } \lambda \in (0,1).$$

Since $||w|| \leq ||y||$ and $\lambda \in (0,1)$ it suffices to show that $-2\langle x, w\rangle \leq -2\langle x, y\rangle + \langle y, y\rangle - \langle w, w\rangle$. But this is equivalent to the assumption that $||x-w|| \leq ||x-y||$. ∎

LEMMA 2.6: *Let H be a Hilbert space. Then M is a weakly closed Chebyshev subset of H if and only if for every $z \in M$ and every $r > 0$ the set $B(z,r) \cap M$ is a weakly compact Chebyshev subset of H.*

Proof: Let M be a weakly closed Chebyshev subset of H, $z \in M$ and $r > 0$. Evidently $B(z,r) \cap M$ is weakly compact. It is also easy to prove that $P_M\colon H \to M$ is norm-norm continuous. Indeed, if $(x_n)_n$ is a sequence in H which strongly converges to a $x \in H$, then since M is boundedly weakly compact it can be proven (as in the case where M is weakly compact) that $(P_M(x_n))_n$ weakly converges to $P_M(x)$ and $\|x - P_M(x_n)\| \to \|x - P_M(x)\|$. But then $(P_m(x_n))_n$ strongly converges to $P_M(x)$.

We can suppose that $z = 0$. Let $x \in H$. If $P_M(x) \in B(0,r) \cap M$ then $P_M(x)$ is the unique point of $B(0,r) \cap M$ that is nearest to x. If $P_M(x) \notin B(0,r) \cap M$ then $\|P_M(x)\| > r$. We set $\Lambda_x = \{\lambda \in [0,1] : \|P_M(\lambda x)\| = r\}$. Since P_M is norm-norm continuous, $P_M(0) = 0$ and $\|P_M(x)\| > r$; the set Λ_x is a non-empty closed subset of $[0,1]$. Let $\lambda_0 = \min \Lambda_x$. Then $\|P_M(\lambda_0 x)\| = r$ and, by Lemma 2.5, we have that $B(x, \|x - P_M(\lambda_0 x)\|) \cap (B(0,r) \cap M) \subset B(\lambda_0 x, \|\lambda_0 x - P_M(\lambda_0 x)\|) \cap (B(0,r) \cap M) = \{P_M(\lambda_0 x)\}$. This means that $P_M(\lambda_0 x)$ is the unique point of $B(0,r) \cap M$ which is nearest to x. Therefore $B(0,r) \cap M$ is a weakly compact Chebyshev subset of H.

To prove the converse let $M \subset H$ such that $B(z,r) \cap M$ is a weakly compact Chebyshev subset of H for every $z \in M$, $r > 0$. Obviously M is weakly closed. We can suppose that $0 \in M$ and $z = 0$. Let $x \in H \setminus \{0\}$. We set $r = 2\|x\|$ and $M_r = B(0,r) \cap M$. Since $\|x - P_{M_r}(x)\| \leq \|x - 0\| = \|x\|$, $B(x, \|x - P_{M_r}(x)\|) \subset B(0,r)$. So $B(x, \|x - P_{M_r}(x)\|) \cap M = B(x, \|x - P_{M_r}(x)\|) \cap B(0,r) \cap M = \{P_{M_r}(x)\}$. That is, $P_{M_r}(x)$ is the unique point of M which is nearest to x. ∎

The proof of Theorem 1.3 follows immediately from Lemma 2.6 and Theorem 1.1. ∎

NOTE 1: Using a similar proof to that of Theorem 1.1 the following can be proven:

THEOREM 2.7: *If X is an almost smooth dual space, then every weakly* compact Chebyshev subset of X is convex.*

NOTE 2: A point x of a closed convex set L is an n-**denting** point of L if: (a) x is an n-extreme point of L and (b) x is a point of continuity of the identity map id_L: $(L, weak) \to (L, norm)$. In [12] it was proved that if the set L has the RNP, then it is the weak closure of its n-denting points. Using exactly the same methods, the notion of n-denting points in place of n-extreme points and the Troyanski–Lin Lemma in place of Choquet's Lemma, the following result can be proven:

THEOREM 2.8: *Let X be an almost smooth Banach space and M a bounded Chebyshev subset of X such that the restriction of the projection $P_M\colon X \to M$ in every finite dimensional subspace of X is norm-weak continuous and the set $L = \overline{\mathrm{conv}}\,(M)$ has the RNP. Then $\overline{M}^w = L$, that is, $\overline{M}^w$ is convex.*

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